Generalisation of the test theory of special relativity to non-inertial frames

1989 J. Phys. A: Math. Gen. 22 1589

(http://iopscience.iop.org/0305-4470/22/10/014)

J. Phys. A: Math. Gen. 22 (1989) 1589-1597. Printed in the UK

Generalisation of the test theory of special relativity to non-inertial frames

G H Abolghasem†, M R H Khajehpour‡§ and R Mansouri‡§

† Department of Physics, Sharif University of Technology, Tehran, Iran
‡ International Centre for Theoretical Physics, Miramare, Trieste, Italy

Received 5 October 1986

Abstract. We present a generalised test theory of special relativity, using a non-inertial frame. Within the framework of the special theory of relativity the transport and Einstein synchronisations are equivalent on a rigidly rotating disc. But in any theory with a preferred frame, such an equivalence does not hold. The time difference resulting from the two synchronisation procedures is a measurable quantity within the reach of existing clock systems on the Earth. The final result contains a term which depends on the angular velocity of the rotating system, and hence measures an absolute effect. This term is of crucial importance in our test theory of special relativity.

1. Introduction

In the theory of relativity, one usually synchronises the clocks by the so-called Einstein procedure using light signals. An alternative procedure, which has also been widely discussed, is the synchronisation by 'slow transport' of clocks. The equivalence of the two procedures in inertial frames was first shown by Eddington (1963). In non-inertial frames the problem is more subtle and has recently been the subject of several articles, with conflicting results. Cohen *et al* (1983) have claimed that the two synchronisation methods are not equivalent, while Ashby and Allan (1984) have made a careful analysis of the problem and have established the equivalence of the two procedures. To realise the practical importance of this problem, one need only consider the degree of accuracy in time and frequency measurements obtained in the last two decades (10^{-9} s) (see Ashby and Allan 1984, Allan 1988).

In this paper we approach the problem in the framework of the test theory of special relativity suggested by Mansouri and Sexl (1977) (see also Mansouri 1988). In this test theory, a class of rival theories is introduced against which the special theory of relativity is tested. The rival theories are all theories with a preferred frame of reference ('ether' or stationary frame) and for them the two methods of synchronisation are not, in general, equivalent. The transformation from the ether frame to any other frame moving with a constant velocity (inertial frame) is a linear expression with a set of velocity-dependent coefficients and to its time component we can always add a synchronisation term by convention, which is an arbitrary function of space, time and velocity. Special relativity is recovered when these coefficients adopt a specific set of values. The time difference resulting from the two synchronisation procedures

§ Permanent address: Department of Physics, Sharif University of Technology, Tehran, Iran.

provides us with a means, either theoretical or experimental, to set limits on the accuracy of the special theory of relativity.

Here we try to extend the above scheme to non-inertial frames such as a rotating disc. In § 2 we review the synchronisation procedures for accelerated observers in the special theory of relativity. It is well known that the transformation from a stationary frame to a rigid rotating frame causes the velocity of light to be locally different in opposite directions. One should redefine time, or rather 'correct' the local time interval of two adjacent events by an amount δ_E, so that the speed of light becomes the same in both directions. This corrected, or 'natural time', interval (Tonnelat 1966) guarantees the clocks on the rotating disc to be Einstein-synchronised. Since the classical Sagnac effect is related to this 'correction', we may call this redefinition of time interval the 'local Sagnac effect'. On the other hand, the synchronisation by slow clock transport on the rotating disc requires another correction in local time interval ('local transport Sagnac effect') to arrive at the 'proper time' of the transported clock. Therefore, several time intervals are involved in the problem, which require some elucidation if the equivalence of the two synchronisation procedures is to be investigated. In practice, to establish a network of synchronised clocks on, or in orbit around, the rotating Earth one needs a clear picture of the situation involved. We believe that the lack of differentiation between these time intervals has been the source of contradictory statements in the literature. In § 3 we consider a rotating system in our ether frame and evaluate the above-mentioned corrections using a particular one-parameter test theory and show that the difference of the corrections is of first order in Ω, the angular velocity of the rotating system. A more general situation discussed in § 3 is when the axis of the rotating system itself is moving with a constant velocity in the ether frame. The application to experiments done on the Earth and the upper limit obtained for the parameter of the theory is discussed in § 4.

2. Synchronisation on the rotating disc

Consider a disc rotating with angular velocity Ω relative to the stationary frame S_0. The clocks in S_0 are all Einstein-synchronised and we have for the metric

$$ds^2 = dt_0^2 - d\boldsymbol{x}_0^2 \tag{2.1}$$

where t_0 and $\boldsymbol{x}_0$ are the temporal and spatial coordinates in S_0. At any moment t, we can attribute a Lorentz frame S_x to a point $\boldsymbol{x}$ on the rigid rotating disc, moving with velocity $\boldsymbol{w}$ of that point in the stationary frame S_0. The Lorentz transformation between S_0 and S_x is given by

$$\begin{aligned} t_0 &= (t + \boldsymbol{w}\cdot\boldsymbol{x})\gamma \\ \boldsymbol{x}_0 &= \boldsymbol{x} - \boldsymbol{w}\left((1-\gamma)\frac{\boldsymbol{w}\cdot\boldsymbol{x}}{w^2} - \gamma t\right). \end{aligned} \tag{2.2}$$

In the frame S_x, at the moment t, we have $\boldsymbol{w}\cdot\boldsymbol{x} = 0$. Then

$$\begin{aligned} t_0 &= \gamma t \\ \boldsymbol{x}_0 &= \boldsymbol{x} + \gamma \boldsymbol{w} t. \end{aligned} \tag{2.3}$$

This transformation actually relates the stationary frame to an infinity of inertial frames instantaneously at rest on the disc at $|\boldsymbol{x}| = r = \text{constant}$. The condition $\boldsymbol{w}\cdot\boldsymbol{x} = 0$

relates the systems S_x to each other and brings us to the rigidly rotating disc. The clocks on the disc showing time t (local time) are externally and absolutely synchronised. Under (2.3) the spacetime separation of two events $(t_0, \boldsymbol{x}_0)$ and $(t_0+\mathrm{d}t_0, \boldsymbol{x}_0+\mathrm{d}\boldsymbol{x}_0)$ is given by

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{d}t_0^2 - \mathrm{d}\boldsymbol{x}_0^2 &= \mathrm{d}t^2 - \frac{2\Omega r^2}{(1-\Omega^2 r^2)^{1/2}}\,\mathrm{d}\varphi\,\mathrm{d}t - r^2\,\mathrm{d}\varphi^2 \\ &= g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \end{aligned} \tag{2.4}$$

where φ is the polar angle on the disc.

It is obvious that the Einstein-synchronisation of the S_0 frame is not preserved on the disc. In order to obtain the same velocity of light in both directions one should 'correct' the time interval $\mathrm{d}t$ by an amount (Landau and Lifshitz 1975)

$$\begin{aligned} \delta_{\mathrm{E}} &= -\frac{g_{0\alpha}}{g_{00}}\,\mathrm{d}x^\alpha \\ &= \frac{\Omega r^2}{(1-\Omega^2 r^2)^{1/2}}\,\mathrm{d}\varphi. \end{aligned} \tag{2.5}$$

Then

$$\mathrm{d}t_{\mathrm{E}} = \mathrm{d}t - \frac{\Omega r^2}{(1-\Omega^2 r^2)^{1/2}}\,\mathrm{d}\varphi \tag{2.6}$$

where t_{E} is the Einstein-synchronised time on the disc. Integrating over a closed circle of radius $|\boldsymbol{x}| = r$, the correction would amount to

$$\Delta_{\mathrm{E}} = \pm\frac{2\pi r^2\Omega}{(1-\Omega^2 r^2)^{1/2}} \tag{2.7}$$

where the signs $\pm$ depend on the direction the circuit is traversed. Two light rays leaving a point A on the disc in opposite directions and traversing the same circular (or polygonal) path would have a phase difference on arriving at A which corresponds to the time difference $2\Delta_{\mathrm{E}}$. This is the well known Sagnac effect (Post 1967).

Now consider a clock moving slowly on a circular path of radius r on the disc. The proper time interval $\mathrm{d}\tau$ shown on the transported clock is given by

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 = \mathrm{d}t^2 - \frac{2\Omega r^2}{(1-\Omega^2 r^2)^{1/2}}\,\mathrm{d}\varphi\,\mathrm{d}t - r^2\,\mathrm{d}\varphi^2. \tag{2.8}$$

$\mathrm{d}\varphi$ is the angle travelled by the clock on the disc in local time $\mathrm{d}t$. By slow clock transport we mean

$$r\,\mathrm{d}\varphi/\mathrm{d}t \ll 1. \tag{2.9}$$

Keeping only the first-order term in $\mathrm{d}\varphi/\mathrm{d}t$, (2.8) leads to

$$\mathrm{d}\tau = \mathrm{d}t - \delta_{\mathrm{T}} \tag{2.10}$$

where

$$\delta_{\mathrm{T}} = \frac{\Omega r^2}{(1-\Omega^2 r^2)^{1/2}}\,\mathrm{d}\varphi. \tag{2.11}$$

Equations (2.5) and (2.11) show that local time interval should be 'corrected' by the same amount, no matter which synchronisation procedure is adopted. This proves the local equivalence of the two methods of synchronisation.

In general, $\mathrm{d}t_{\mathrm{E}}$ and $\mathrm{d}\tau$ are inexact differentials, i.e. their integrals are path dependent. Therefore, in order to have synchronised clocks at two arbitrary points on the disc, the light ray and transported clock should follow the same path between the points.

Now we consider the differential form of (2.3) and write it in the following form:

$$\begin{aligned} &\mathrm{d}t_0 = \Gamma\,\mathrm{d}t \\ &\mathrm{d}r_0 = \mathrm{d}r \qquad |\boldsymbol{x}_0| = |\boldsymbol{x}| = r \\ &\mathrm{d}\varphi = \mathrm{d}\varphi + \Gamma\Omega\,\mathrm{d}t \end{aligned} \tag{2.12}$$

where Γ is now an undetermined factor depending on $w = \Omega r$. Equation (2.12) still converts the line elements $\mathrm{d}s^2 = \mathrm{d}t_0^2 - \mathrm{d}\boldsymbol{x}_0^2$ into the following form for the rotating frame:

$$\mathrm{d}s^2 = \Gamma^2(1 - r^2\Omega^2)\,\mathrm{d}t^2 - 2r^2\Gamma\Omega\,\mathrm{d}\varphi\,\mathrm{d}t - r^2\,\mathrm{d}\varphi^2. \tag{2.13}$$

To have Einstein-synchronisation of the adjacent clocks on the disc (which are not in the same inertial frames), one should correct the time interval $\mathrm{d}t$ such that

$$\mathrm{d}t_{\mathrm{E}} = \mathrm{d}t - \frac{r^2\Omega}{\Gamma(1 - r^2\Omega^2)}\,\mathrm{d}\varphi. \tag{2.14}$$

One can show that the proper time of the moving clock on the disc ($r = \text{constant}$) is equal to $\mathrm{d}t_{\mathrm{E}}$ if and only if

$$\Gamma = (1 - r^2\Omega^2)^{-1/2} = \gamma$$

i.e. the rates of clocks on the disc obey the Lorentz dilation.

If $\Gamma \neq \gamma$, then the two synchronisation methods are not equivalent. Therefore (2.12) with Γ undetermined provides us with a one-parameter test theory of special relativity. In the following section we consider a slightly different rival theory for special relativity which is more appropriate for our purpose.

3. Formulation of the test theory on a rotating disc

Let S_0 again be a preferred stationary frame with coordinates $(t_0, \boldsymbol{x}_0)$. The most general transformation from S_0 to a frame S_x moving with velocity $\boldsymbol{w}$ is given by (Mansouri and Sexl 1977)

$$\begin{aligned} t_0 &= \frac{1}{a} - \frac{\boldsymbol{\varepsilon}\cdot\boldsymbol{x}}{a} \\ \boldsymbol{x}_0 &= \frac{\boldsymbol{x}}{d} + \frac{1}{a}\,\boldsymbol{w}t - \left(\frac{b-d}{bd}\,\frac{1}{w^2}\,\boldsymbol{w}\cdot\boldsymbol{x} + \frac{1}{a}\,\boldsymbol{\varepsilon}\cdot\boldsymbol{x}\right)\boldsymbol{w} \end{aligned} \tag{3.1}$$

where a, b and d are parameters that may depend only on w^2 and $\boldsymbol{\varepsilon}$ is a w-dependent vector parameter specifying the synchronisation procedure. For the Lorentz transformation, i.e. no preferred frame and Einstein-synchronisation, we have

$$a = \gamma^{-1}(w) \qquad b = \gamma(w) \qquad d = 1 \qquad \boldsymbol{\varepsilon} = -\boldsymbol{w}.$$

To have Einstein-synchronisation in the S_x frame one should have (Mansouri and Sexl 1977)

$$\varepsilon = -\frac{a(w)}{b(w)}\frac{w}{1-w^2}. \tag{3.2}$$

In order to obtain the transformation for the disc we should impose the condition $\boldsymbol{x}\cdot\boldsymbol{w}=0$. Then, assuming (3.2),

$$\begin{aligned} t_0 &= t/a \\ \boldsymbol{x}_0 &= \boldsymbol{x}/d + (1/a)\boldsymbol{w}t. \end{aligned} \tag{3.3}$$

For differentials

$$\begin{aligned} \mathrm{d}t_0 &= (1/a)\,\mathrm{d}t \\ \mathrm{d}\boldsymbol{x}_0 &= (1/d)\,\mathrm{d}\boldsymbol{x} + (1/a)\boldsymbol{w}\,\mathrm{d}t \end{aligned} \tag{3.4}$$

and

$$|\mathrm{d}\boldsymbol{x}_0| = r\,\mathrm{d}\Phi \qquad |\mathrm{d}\boldsymbol{x}| = r\,\mathrm{d}\varphi \qquad w = r\Omega.$$

The adjacent observers on the disc with $|\boldsymbol{x}| =$ constant do not belong to the same inertial frame. Therefore, as it stands, their clocks are not Einstein-synchronised with each other:

$$\mathrm{d}t_0^2 - \mathrm{d}\boldsymbol{x}_0^2 = (1/a)^2\,\mathrm{d}t^2 - [(1/d)\,\mathrm{d}\boldsymbol{x} + (1/a)\boldsymbol{w}\,\mathrm{d}t]^2. \tag{3.5}$$

To achieve such a synchronisation on the disc, we define a new time interval $\mathrm{d}t_{\mathrm{E}}$:

$$\mathrm{d}t_{\mathrm{E}} = \mathrm{d}t - (a/d)\frac{w}{1-w^2}\,r\,\mathrm{d}\varphi. \tag{3.6}$$

Then

$$\mathrm{d}t_0^2 - \mathrm{d}\boldsymbol{x}_0^2 = [(1-w^2)/a^2]\,\mathrm{d}t_{\mathrm{E}}^2 - (1/d)^2[1/(1-w^2)]r^2\,\mathrm{d}\varphi^2. \tag{3.7}$$

Equation (3.7) can be written as

$$\mathrm{d}s^2 = \mathrm{d}t_0^2 - \mathrm{d}\boldsymbol{x}_0^2 = f^2(w)(\mathrm{d}t_{\mathrm{E}}^2 - K^2 r^2\,\mathrm{d}\varphi^2) \tag{3.8}$$

where

$$f(w) = \frac{(1-w^2)^{1/2}}{a(w)} \qquad K(w) = \frac{a(w)}{d(w)}\frac{1}{1-w^2} \tag{3.9}$$

and $f(w)$ and $K(w)$ are functions of w^2 only.

One should note that, in contrast to the special theory of relativity, s no longer represents the proper time of clocks located on the disc. This time is already represented by t_{E}, which differs from s as long as $f(w) \neq 1$. Here with the metric given by (3.8) we have the preferred frame corresponding to $w = 0$. No coordinate transformation can reduce $f(w)$ to unity when $w \neq 0$. One can, of course, rescale the rates of clocks and the standard of length in S_x by the same velocity-dependent factor $f(w)$ to achieve the equivalence of Einstein and transport synchronisations. However, bringing these rescaled clocks and standards of length back to the ether frame S_0, the equivalence of Einstein and transport synchronisations would break down.

Now we try to find the difference of the corrections, $|\delta_E - \delta_T|$. To each point on the disc (r = constant) we attach a frame S_x moving with speed w. We assume that all the clocks on the ring r = constant are Einstein-synchronised. Now, consider a clock moving on this same ring with speed w' relative to S_0. Then by (3.8) we have

$$dt_0^2 - dx_0^2 = f^2(w')\, d\tau^2 \tag{3.10}$$

where τ is the time measured by the moving clock (proper time of the clock) and w' is the speed of the moving clock relative to the stationary system S_0. From (3.8) and (3.10) we obtain

$$d\tau = \frac{f(w)}{f(w')}(1 - K^2(w)u^2)^{1/2}\, dt_E \tag{3.11}$$

where $u = r(d\varphi/dt_E)$ is the speed of the moving clock relative to the disc. For small u values (A3)

$$w' \simeq \frac{a(w)}{b(w)} u + w \tag{3.12}$$

and then

$$f(w') - f(w) \simeq \frac{a(w)}{b(w)} u \frac{df(w)}{dw}. \tag{3.13}$$

Now we expand f up to first order in w^2:

$$f(w) \simeq 1 - \tfrac{1}{2}(2\alpha + 1)w^2.$$

Then at any point on the disc, the proper time of the clock transported a finite angle $\Delta\varphi$ would differ from the time shown by the Einstein-synchronised clocks, up to linear terms in the velocity by an amount (Abolghasem *et al* 1988)

$$|\Delta t_E - \Delta\tau| \simeq (2\alpha + 1) w\, \Delta\varphi. \tag{3.14}$$

When no inertial frame is preferred (special relativity) $\alpha = -\frac{1}{2}$ and $f(w) = 1$, i.e. $\Delta t_E = \Delta\tau$. This proves the equivalence of the two synchronisation methods on a rotating disc in the framework of special relativity.

In the above derivation we made the simplifying assumption that the axis of rotation is at rest in the ether frame. This assumption, while quite helpful for the introduction of different 'times' and 'time corrections', is not applicable to systems such as the moving Earth. We may consider the local frame of cosmic background radiation as the stationary ether frame, relative to which the Earth is moving with a speed of 300 km s^{-1}. Therefore, a hierarchy of four coordinate systems are to be taken into account: a preferred 'ether' frame Σ, a non-rotating frame S_0 moving in Σ, a rigidly rotating system S_x spinning with angular velocity Ω in S_0, and finally a clock moving slowly in S_x at a fixed distance from the axis of rotation. Our parametrised rival theories provide us with a set of transformations relating each of the above reference systems to Σ. One should find the relative velocity of S_x and S_0, impose the condition of rigid rotation and calculate the 'Sagnac type' and 'transport Sagnac type' time corrections, just as we did before. The details of these calculations can be found in the appendix. The final result (A14) is very similar to (3.14):

$$\Delta t_E - \Delta t_T = (1 + 2\alpha) Vr\, \Delta\varphi \tag{3.15}$$

or using (A4)

$$\Delta t_E - \Delta t_T \approx (1 + 2\alpha)\left(1 + \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{v^2}\right) vr\, \Delta\varphi \tag{3.16}$$

where V is the speed of S_x relative to Σ and $\Delta\varphi$ is the angular displacement of the transported clock on the disc. w is the relative velocity of S_x and S_0 as before. When $v=0$, (3.15) is reduced to (3.14) by way of (A3). It should be noted that (3.14) is not recovered when v is set equal to zero in (3.16). For $w=0$ it can easily be seen that the previously obtained result of Mansouri and Sexl (1977) is regained.

4. Conclusion

Equivalence of the two synchronisation methods in the special theory of relativity implies that $\alpha=-\frac{1}{2}$. Therefore (3.16) provides us with a means for testing this theory against a class of rival theories defined by a parameter α. Before discussing the implications for the experimental results we should note an important point. The right-hand side of (3.16) consists of two terms which are fundamentally different in nature. The first term, linear in v, is in a sense a relative quantity, depending on which frame is taken as preferred. The second term, related to the Sagnac effect, depends on the angular velocity of the rotating system. Hence it measures an absolute effect, independent of the choice of the ether frame. This makes the second term of crucial importance in our test theory, even though numerically it may be much smaller than the first term.

All of the so-called first-order experiments of the special theory of relativity are related to the results obtained above. In these experiments the one-way speed of light is measured using distant transport-synchronised clocks. We divide these experiments into two groups.

(i) Laboratory experiments where the quantity $r\Delta\varphi$ in (3.16) is of the order of 1m. This group includes the older rotor tests (Champaney *et al* 1963, Isaak 1970), as well as the recent work of Riis *et al* (1988) in which the isotropy of the speed of light is tested. For these experiments the second term of (3.16) is negligibly small compared to the first term $(10^{-15}/10^{-8})$. The experimental equality of Δt_E and Δt_T within the limits of accuracy of existing atomic clocks (a few nanoseconds, Asby and Allan (1984)) results in $\alpha=-\frac{1}{2}\pm 10^{-5}$, if we consider the local frame of microwave background radiation as our ether frame.

(ii) The second group of experiments are large-scale tests such as the around-the-world clock-transport experiments where $r\Delta\varphi$ is of order of 10^7m. Here the second term of (3.16) is no longer negligible. The data from GPS satellites with their much larger $r(\sim 20\text{-}25\times 10^3\,\text{km})$ may bring this term well within the range of accuracy of existing atomic clocks (Allan 1988). Considering the absolute nature of the second term in (3.16), it is desirable to arrange an experimental set-up in which the first term is somehow eliminated. Vanishing of the second term along a certain meridian is helpful in this respect. One possible arrangement is a null first-order experiment performed on Earth using a clock first transported along the meridian and then along some other direction, preferably parallel to the equator.

Acknowledgments

Two of us (MRHK and RM) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Miramare, Trieste, Italy. RM would also like to thank

Professors D W Allan, B Bertotti and O Poulsen for private communications and helpful discussions.

Appendix

Let us denote the 'ether' frame by $\Sigma(T, X)$. Consider a frame $S_0(t_0, x_0)$ moving with velocity $\boldsymbol{v}$ relative to Σ. If the clocks in S_0 are Einstein-synchronised

$$\varepsilon = -\frac{a(v)}{b(v)}\frac{\boldsymbol{v}}{1-v^2}$$

the general transformations (3.1) can be written as

$$\begin{aligned} \boldsymbol{X} &= \boldsymbol{x}_0 + \frac{1}{a}\boldsymbol{v}t_0 - \left(\frac{b(1-v^2)-1}{bv^2(1-v^2)}\right)(\boldsymbol{v}\cdot\boldsymbol{x}_0)\boldsymbol{v} \\ T &= \frac{t_0}{a} + \frac{1}{b(1-v^2)}\boldsymbol{v}\cdot\boldsymbol{x}_0 \end{aligned} \tag{A1}$$

and their inverse as

$$\begin{aligned} \boldsymbol{x}_0 &= \boldsymbol{X} - b\boldsymbol{v}T + (b-1)\frac{\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{X})}{v^2} \\ t_0 &= \frac{a}{1-v^2}(T - \boldsymbol{v}\cdot\boldsymbol{X}) \end{aligned} \tag{A1$'$}$$

where we have set d equal to unity, as it does not affect the results. For any other system $S_x(t, x)$ moving with velocity $\boldsymbol{V}$ in Σ, a and b are functions of V. The velocity $\boldsymbol{w}$ of S_x relative to S_0 is simply obtained using (A1) and (A1$'$) as

$$\boldsymbol{w} = \frac{\boldsymbol{x}_0}{t_0}\bigg|_{x=0} = \frac{\boldsymbol{v} - b\boldsymbol{v} + [(b-1)/v^2](\boldsymbol{v}\cdot\boldsymbol{V})\boldsymbol{v}}{[a/(1-v^2)](1-\boldsymbol{v}\cdot\boldsymbol{V})} \tag{A2}$$

where a and b are functions of v. Equation (A2) is nothing but our formula for the addition of velocities. For small values of $w(w^2=0)$, we have

$$\boldsymbol{V} \simeq \boldsymbol{v} + \frac{a(v)}{b(v)}\boldsymbol{w} \tag{A3}$$

$$V \simeq v + \frac{a(v)}{b(v)}\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{v} \tag{A4}$$

and also

$$\begin{aligned} a(V) &\simeq a(v) + \frac{a(v)}{b(v)}\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{v}\frac{\mathrm{d}a(v)}{\mathrm{d}v} \\ b(V) &\simeq b(v) + \frac{a(v)}{b(v)}\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{v}\frac{\mathrm{d}b(v)}{\mathrm{d}v}. \end{aligned} \tag{A5}$$

Now let S_0 be the frame attached to the axis of rotation of the disc (or non-rotating Earth) and S_x be the frame of the rigidly rotating disc (or the Earth). The transformations (A1) and (A1$'$) for S_x is only momentarily defined and should be subject to the condition of rigid rotation:

$$\mathrm{d}|\boldsymbol{x}| = 0 \qquad \boldsymbol{w}\cdot\boldsymbol{x} = 0. \tag{A6}$$

In order to calculate the quantity $\mathrm{d}T^2-\mathrm{d}X^2$, in terms of coordinates (t, x), we make a further approximation and keep the terms only up to second order in v:

$$\begin{aligned} a(v) &\simeq 1+\alpha v^2 \\ b(v) &\simeq 1+\beta v^2. \end{aligned} \tag{A7}$$

Using (A7) and imposing (A6), we obtain

$$\mathrm{d}T^2-\mathrm{d}X^2=f^2(V)\,\mathrm{d}t^2-[1-(2\beta-1)v_\varphi^2 2(\beta-1)vw\sin\varphi]r^2\,\mathrm{d}\varphi^2-2wr\,\mathrm{d}\varphi\,\mathrm{d}t \tag{A8}$$

where $f^2(V)=[1-(1+2\alpha)V^2]$ and φ is the polar angle on the disc. v_φ is the projection of $\boldsymbol{v}$ on $\boldsymbol{w}$. Now

$$\mathrm{d}t_{\mathrm{E}}=\mathrm{d}t-wr\,\mathrm{d}\varphi/f^2(V) \tag{A9}$$

diagonalises the RHS of (A8):

$$\mathrm{d}T^2-\mathrm{d}X^2=f^2(V)(\mathrm{d}t_{\mathrm{E}}^2-K^2r^2\,\mathrm{d}\varphi^2) \tag{A10}$$

where K is a certain function of $\boldsymbol{v}$, $\boldsymbol{w}$ and φ. Now let us denote by S_x' the rest frame of a clock moving at $r=$ constant on the rotating disc. Then

$$\mathrm{d}T^2-\mathrm{d}X^2=f^2(V')\,\mathrm{d}t_{\mathrm{T}}'^2 \tag{A11}$$

where V' is the velocity of S_x' relative to Σ and t_{T}' is the proper time of the clock (i.e. the time shown on the moving clock). One then easily obtains

$$\frac{\mathrm{d}t_{\mathrm{E}}-\mathrm{d}t_{\mathrm{T}}'}{\mathrm{d}t_{\mathrm{E}}}\approx \mathrm{u}\frac{\mathrm{d}f(V')}{\mathrm{d}V'} \tag{A12}$$

where $u=r\,\mathrm{d}\varphi/\mathrm{d}t_{\mathrm{E}}$ is the speed with which the clock moves on the rotating disc S_x. Using the expansion

$$f(V)\simeq 1-\tfrac{1}{2}(1+2\alpha)V^2 \tag{A13}$$

one finds from (A12) that, for a clock transported by a finite angle $\Delta\varphi$,

$$|\Delta t_{\mathrm{E}}-\Delta t_{\mathrm{T}}|=(1+2\alpha)Vr\Delta\varphi. \tag{A14}$$

References

Abolghasem G H, Khajehpour M R H and Mansouri R 1988 *Phys. Lett* A **132** 310
Allan D W 1988 private communication
Ashby N and Allan D W 1984 *Phys. Rev. Lett.* **53** 1858
Champaney D C, Isaak G R and Kahn A M 1963 *Phys. Lett.* 7 241
Cohen J M, Moses H E and Rosenblum A 1983 *Phys. Rev. Lett.* **51** 1501
Eddington A 1963 *The Mathematical Theory of Relativity* (Cambridge: Cambridge University Press) p 15
Issak G R 1970 *Phys. Bull.* **21** 255
Landau L D and Lifshitz E M 1975 *The Classical Theory of Fields* (Oxford: Pergamon) 4th edn
Mansouri R 1988 *Internal Report, ICTP, Trieste* IC/88/55
Mansouri R and Sexl R 1977 *Gen. Rel. Grav.* **8** 497, 515, 809
Post E J 1967 *Rev. Mod. Phys.* **39** 475
Riis E, Anderson L, Bjerre N, Poulsen O, Lee S A and Hall J L 1988 *Phys. Rev. Lett.* **60** 81
Tonnelat M A 1966 *Principles of Electromagnetism and Relativity* (New York: Gordon and Breach)